NPRE-446 Radiation Interaction with Matter I
Homework Assignments

Prof. Yang Zhang

Department of Nuclear, Plasma, and Radiological Engineering
Department of Materials Science and Engineering
Program of Computational Science and Engineering

University of Illinois at Urbana-Champaign

zhyang@illinois.edu

http://zhang.npre.illinois.edu

November 18, 2016

1 Problem Set 1

Explanation of the score or Calories:

Our brains typically consume 0.2 Calories per minute on average. When actively thinking, our brains can kick it up to burning about 1 Calorie per minute. So instead of assigning each question with some Calories, I will assign each question an energy. For example, if a problem is given 10 Calories, it means the problem is worth 10 Calories, or you will need to burn about 10 Calories to solve the problem, or the estimated time to solve the problem is about 10 minutes.

Readings:

Chapter 7, J. R. Taylor, *Classical Mechanics*, University Science Books (2005).

Remarks:

Because of the breath and depth of the content of the course, it is only possible to cover the essence during the lectures. One must read the relevant chapters in the textbooks to learn the details and gain deeper understandings.

1.1

A nucleus, originally at rest, decays radioactively by emitting an electron of momentum 1.73 MeV/c, and at right angles to the direction of the electron a neutrino with momentum 1.00 MeV/c. (The MeV, million electron volt, is a unit of energy used in modern physics, equal to 1.60×10^{-13} J. Correspondingly, MeV/c is a unit of linear momentum equal to 5.34×10^{-22} kg · m/s.) In what direction does the nucleus recoil? What is its momentum in MeV/c? If the mass of the residual nucleus is 3.90 $\times 10^{-25}$ kg what is its kinetic energy, in electron volts? [Goldstein–Poole–Safko: Page 32, Question 1.17] (20 Calories)

1.2

Our sun was born about 5 billion years ago from the gravitational collapse of part of a giant molecular cloud that consisted mostly of hydrogen and helium. In another 5 billion years, the current theory predicts that the sun will transform into a red giant and eventually burns up. One scientist proposes to drill a hole to earth's inner core and then let the sea water in. The emerging jet of steam shall be utilized as rocket drive to move the earth away to another star. How do you judge the proposal? (20 Calories)

Remarks: The temperature of earth's inner core is about $T_e \approx 5000$ K. The radius of the earth is about $R_e \approx 6000$ km. The mass of the earth is about $M_e \approx 6 \times 10^{24}$ kg. The average depth of the sea is $h_{sea} \approx 4$ km. Boltzmann constant $k_B = 1.38 \times 10^{-23}$ J/K.

1.3

In physics, an inverse-square law $\mathbf{F} = -\frac{k}{r^2}\hat{r}$ is ubiquitous, such as Newton's law of universal gravitation and Coulomb's law of electrostatic interaction. To the best of our knowledge, the exponent is exactly 2, which is a consequence of a simple inverse-r potential. In this problem, we will examine the behavior of a fictitious system with a small deviation from the inverse-square law. Assume

$$\mathbf{F} = -\frac{k}{r^{2+\epsilon}}\hat{r}$$

where $k = 1$, ϵ is a very small number. Questions:

1. Is the angular momentum a conserved quantity? Why? (10 Calories)
2. Compute $\nabla \times \mathbf{F}$. (10 Calories)
3. Derive the potential function $V(r)$. (10 Calories)
4. Verify $\mathbf{F} = -\nabla V$. (10 Calories)
5. Consider a particle moving in this force field. Write down the Lagrangian of in polar coordinates (r, θ). Derive the equations of motion. (10 Calories)

1.4

Derive the Lagrangian and the equations of motion for a spherical pendulum, i.e., a mass point m suspended by a rigid weightless rod with length l, in spherical coordinates θ and ϕ. (30 Calories)

2 Problem Set 2

Readings:

Chapter 13, J. R. Taylor, *Classical Mechanics*, University Science Books (2005).

2.1

Regular and chaotic behaviors of a double pendulum.
See http://en.wikipedia.org/wiki/Double_pendulum

1. Choose the angles θ_1 and θ_2 between each limb and the vertical as the generalized coordinates. Derive the equations of motion of a double compound pendulum of two identical uniform rigid rods. (20 Calories)

2. Small oscillations: when both θ_1 and θ_2 are small, solve the above equations of motion analytically and find out the two normal modes and their corresponding normal frequencies. Draw schematically the two normal modes. (20 Calories)

3. Chaotic oscillations: when any of $\theta_1(t=0)$, $\dot{\theta}_1(t=0)$, $\theta_2(t=0)$ and $\dot{\theta}_2(t=0)$ are not small, the equations of motion can only be solved numerically. Read through the wikipedia page to learn about the chaotic motions. (0 Calories)

2.2

The Lagrangian of a charged particle with mass m and charge q in an electromagnetic field is

$$\mathcal{L} = \frac{1}{2}m\dot{\mathbf{r}}^2 - q(\phi - \dot{\mathbf{r}} \cdot \mathbf{A})$$

where ϕ is the electric potential, $\mathbf{A}$ is the magnetic vector potential. Perform Legendre transform to find out the Hamiltonian $\mathcal{H}$. (20 Calories)

2.3

Read Chapter 14 in Taylor or the wikipedia page on Rutherford scattering:
https://en.wikipedia.org/wiki/Rutherford_scattering.
Derive the Rutherford scattering cross section

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z_1 Z_2 e^2}{8\pi\epsilon_0 m v_0^2}\right)^2 \csc^4\frac{\Theta}{2}$$

(60 Calories)

3 Problem Set 3

Readings:

Chapters 1, 2, 5, 7, D. J. Griffiths, *Introduction to Electrodynamics*, 4th edition, Addison-Wesley (2012).

3.1

1. Derive the differential form of the Maxwell's equations using Coulomb's law, Faraday's law, and Biot-Savart law. (40 Calories)

2. Based on symmetry and physical arguments, write down the differential form of the electromagnetic equations if there are magnetic monopoles. Explain the meaning of each equation. (20 Calories)

3.2

Griffiths's Electrodynamics: Page 65, Problem 2.5 (30 Calories)

3.3

Griffiths's Electrodynamics: Page 76, Problem 2.15 (30 Calories)

3.4

If we cut a uniformly charged solid sphere (charge density ρ, radius R) in half, what is the force of repulsion between the two hemispheres? (30 Calories)

4 Problem Set 4

Readings:

Chapter 3, D. J. Griffiths, *Introduction to Electrodynamics*, 4th edition, Addison-Wesley (2012).

4.1

Considering the energy conservation

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{f} \cdot \mathbf{v}$$

where $\mathbf{f}$ is the Lorentz force density $\mathbf{f} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}$, derive the expressions of the energy density u and the Poynting vector $\mathbf{S}$. (15 Calories)

4.2

A point charge q is placed a distance d above an infinite grounded conducting plane (see Fig. 3.10 on page 125 and section 3.2.1 to 3.2.3 in Griffiths's Electrodynamics).

1. What is the potential above the plane? What is the potential below the plane? (5 Calories)
2. What is the induced surface charge distribution? What is the total induced charge? (5 Calories)
3. What is the force exerted on the point charge? (5 Calories)
4. How much energy does it take to move the point charge to infinity? (5 Calories)

4.3

Griffiths's Electrodynamics: Page 128, Question 3.2 (20 Calories)

4.4

Griffiths's Electrodynamics: Page 130, Question 3.11 (30 Calories)

5 Problem Set 5

Readings:

Chapter 9, D. J. Griffiths, *Introduction to Electrodynamics*, 4th edition, Addison-Wesley (2012).

5.1

Griffiths's Electrodynamics: Page 432, Question 9.35 (45 Calories)

5.2

Griffiths's Electrodynamics: Page 400, Question 9.10 (20 Calories)

5.3

Griffiths's Electrodynamics: Page 415, Question 9.19 (30 Calories)

5.4

1. Why does water appear transparent to our eyes? Explain the physics behind the phenomenon. Name a few more particles (at least three) for which water is not transparent, and explain why. (10 Calories)

2. Why do newly-polished metals show shiny lustrous appearance? Why do some metals (e.g., Fe, Cu) lose the luster very quickly over time, and some (e.g., Au, stainless steel) slowly? Explain the physics behind the phenomenon. (10 Calories)

3. What is the color of a typical semiconductor, such as Si? Explain the physics behind the color. Note that the band gap of Si is about 1.1 eV. (10 Calories)

6 Problem Set 6

Readings:

Chapters 10, 11, D. J. Griffiths, *Introduction to Electrodynamics*, 4th edition, Addison-Wesley (2012).

6.1

Griffiths's Electrodynamics: Page 448, Question 10.11 (30 Calories)

6.2

An electron is released from the roof of Talbot Lab and falls under the influence of gravity only. When the electron hits the ground, what fraction of the potential energy is radiated? Assume the height of Talbot Lab is 30 m. A valid answer is an estimate of the order of magnitude. (20 Calories)

Notes: Larmor formula

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

$m_e = 9.11 \times 10^{-31}$ kg
$q_e = 1.6 \times 10^{-19}$ C
$\mu_0 = 4\pi \times 10^{-7}$ N/A^2
$g = 9.8$ m/s^2
$c = 3.0 \times 10^8$ m/s.

6.3

Griffiths's Electrodynamics: Page 485, Example 11.3, bremsstrahlung radiation (40 Calories)

6.4

Griffiths's Electrodynamics: Page 487, Question 11.16, synchrotron radiation (40 Calories)

7 Problem Set 7

Readings:

Chapter 1, D. J. Griffiths, *Introduction to Quantum Mechanics*, 2nd edition, Pearson Prentice Hall (2004).

7.1

1. Explain Compton Scattering. (10 Calories)
2. Compute the Compton shift formula:

$$\Delta\lambda = \lambda' - \lambda = \lambda_c(1 - \cos\theta)$$

where $\lambda_c = \frac{2\pi\hbar}{m_e c}$ is the Compton wavelength of an electron. (30 Calories)

7.2

Griffiths's Quantum Mechanics: Page 20, Problem 1.10. (20 Calories)

7.3

Griffiths's Quantum Mechanics: Page 20, Problem 1.9. (30 Calories)

7.4

Griffiths's Quantum Mechanics: Page 22, Problem 1.17. (30 Calories)

7.5

Griffiths's Quantum Mechanics: Page 23, Problem 1.18. (30 Calories)

8 Problem Set 8

Readings:

Chapter 2, D. J. Griffiths, *Introduction to Quantum Mechanics*, 2nd edition, Pearson Prentice Hall (2004).

8.1

Consider a particle in a one dimentional infinite square well potential:

$$V(x) = \begin{cases} 0, & 0 \le x \le a \\ \infty, & \text{otherwise} \end{cases}$$

1. Compute the eigen energies and the eigen state wave functions. (20 Calories)
2. Compute $\langle x \rangle$, $\langle x^2 \rangle$, σ_x, $\langle p \rangle$, $\langle p^2 \rangle$, σ_p for the n-th eigen state. Compute the uncertainty relation quantity $\sigma_x \sigma_p$. Check whether the uncertainty relation is satisfied. Which state is closest to the uncertainty limit? (20 Calories)
3. If the initial wave function is

$$\Psi(x, 0) = \begin{cases} Ax, & 0 \le x \le \frac{a}{2} \\ A(a - x), & \frac{a}{2} \le x \le a \\ 0, & \text{otherwise} \end{cases}$$

   for some constant A.

   (a) Sketch $\Psi(x, 0)$. Determine the constant A. (10 Calories)
   (b) Compute $\Psi(x, t)$. (20 Calories)
   (c) If we perform a measurement of the energy, what shall we get? (10 Calories)

8.2

Griffiths's Quantum Mechanics: Page 38, Problem 2.5. (30 Calories)

9 Problem Set 9

Readings:

Chapter 2, D. J. Griffiths, *Introduction to Quantum Mechanics*, 2nd edition, Pearson Prentice Hall (2004).

9.1

The Hamiltonian of a 1-dimensional quantum harmonic oscillator is

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

Its eigen wave functions $\psi_n(x)$ and the corresponding eigen energy E_n are

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{\lambda}{\pi}\right)^{1/4} \exp\left(-\frac{1}{2}\lambda x^2\right) H_n\left(\sqrt{\lambda}x\right)$$

$$E_n = (n + \frac{1}{2})\hbar\omega$$

where $\lambda = m\omega/\hbar$, $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$ are the Hermite polynomials.

1. For $n = 0, 1, 2$, find out and sketch the wave functions $\psi_n(x)$ and the square of the wave functions $|\psi_n(x)|^2$ on top of the potential. (10 Calories)
2. For $n = 0, 1, 2$, check the orthogonality of $\psi_n(x)$ by explicit integration. (15 Calories)
3. For $n = 0, 1, 2$, compute $\langle x \rangle$, $\langle x^2 \rangle$, σ_x, $\langle p \rangle$, $\langle p^2 \rangle$, σ_p. Compute the uncertainty relation quantity $\sigma_x\sigma_p$. Check whether the uncertainty relation is satisfied. Which state is closest to the uncertainty limit? (15 Calories)
4. For $n = 0, 1, 2$, compute the expectation values of the kinetic energy $\langle T \rangle$ and the potential energy $\langle V \rangle$. (15 Calories)
5. If the particle starts out in the initial state

   $$\Psi(x, 0) = A\left[3\psi_0(x) + 4\psi_1(x)\right]$$

   what is A? what is $\Psi(x, t)$? If we measure the energy of this particle, what values may we get and with what probabilities? (15 Calories)

9.2

Griffiths's Quantum Mechanics: Page 83, Problem 2.34. (40 Calories)

9.3

Griffiths's Quantum Mechanics: Page 84, Problem 2.35. (30 Calories)

10 Problem Set 10

Readings:

Chapter 2, D. J. Griffiths, *Introduction to Quantum Mechanics*, 2nd edition, Pearson Prentice Hall (2004).

10.1

1. From the continuity equation
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$
derive the probability current
$$\mathbf{J} = \frac{i\hbar}{2m}(\Psi \nabla \Psi^* - \Psi^* \nabla \Psi) = \frac{\hbar}{m}\mathrm{Im}\{\Psi^* \nabla \Psi\}$$
(10 Calories)

2. Compute the probability current for the plain wave $\Psi(x,t) = e^{i(kx-\omega t)}$. (10 Calories)

3. Compute the probability current for the spherical wave $\Psi(\mathbf{r},t) = e^{i(\mathbf{kr}-\omega t)}/\mathbf{r}$. (10 Calories)

10.2

In atomic and nuclear physics, many systems can be described by pseudo-potentials, such as the Fermi pseudo-potential $V(r) = \frac{2\pi\hbar^2}{m} b\delta(r)$ used in describing neutron scattering processes, where b is the bound scattering length. Let's consider a particle with mass m in the 1-dimensional δ-potential

$$V(x) = -V_0 \delta(x)$$

where $V_0 > 0$ is the strength of the potential with the dimension of [EL].

1. Compute the bound state ($E < 0$) wave function and the corresponding eigen energy level(s). (30 Calories)

2. Compute the probability current of the wave function in Part 1. Explain the physical meaning of the result. (5 Calories)

3. Compute the transmitted and reflected wave functions of an incoming plane wave $\psi_i(x) = Ae^{ikx}$ ($E > 0$, scattering state). (30 Calories)

4. Compute the probability current on both sides of the potential in Part 3. Explain the physical meaning of the result. (10 Calories)

5. Compute the transmission coefficient T and reflection coefficient R in Part 3. (10 Calories)

6. Perform dimensional analysis on the eigen energy computed in Part 1, and the transmission and reflection coefficients computed in Part 5. (10 Calories)

7. What's the asymptotic behavior of T and R if the potential is very deep, i.e. $V_0 \to \infty$? Explain the physical meaning. (5 Calories)

10.3

Macroscopic quantum world: Planck constant (reduced) $\hbar \approx 1 \times 10^{-34} J \cdot s$ plays a fundamental role in quantum mechanics. Imagine that one day you are transported to another universe, where the reduced Planck constant is 10^{34} times larger, i.e. $\hbar \approx 1 J \cdot s$. Let's picture what strange phenomena you would expect.

1. First, derive the uncertainty principle

$$\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle \geq \frac{1}{4}\left|\langle[\hat{A},\hat{B}]\rangle\right|^2$$

   and state its the significance. (10 Calories)

2. Suppose you have a jar of candies in this quantum world. When you open the jar, estimate the escape velocities of the candies. Do you need to be careful when opening the jar? (A typical weight of a candy is 1 gram. A typical size of a jar is 10 cm.) (10 Calories)

3. Use your imagination, describe another two strange phenomena you would expect in this quantum world. Be as quantitative as possible. (20 Calories)