NPRE 447 & 521 Interaction of Radiation with Matter II

Homework Assignments

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1 Homework 1

Attention:

1. Both 447 and 521 students should solve the problems without asterisks. In addition, 521 students should also solve the problems marked with asterisks. 447 students will not get extra credit for solving the problems with asterisks.
2. Write down your UIN instead of your name if you worry about privacy when turning in homework. Also write down the course number next to your UIN clearly.
3. Explanation of the score: our brains typically consume about 0.2 Calories per minute. When actively thinking, our brains can kick it up to burning about 1 Calorie per minute. So instead of assigning each question with points, I will assign with Calories. For example, if a problem is given 10 Calories, it means you will need to burn about 10 Calories to solve the problem and the estimated time to solve the problem is about 10 minutes.
4. Because of the breath and depth of the content of the course, it is only possible to cover the essence during the lectures. One must read the relevant chapters in the textbooks to learn the details and gain deeper understandings.

Readings:

Chapter 1 and 2, D. J. Griffiths, *Introduction to Quantum Mechanics*, 2nd edition, Pearson Prentice Hall (2004).

1.1

In Bohr's theory of hydrogen, the electron in its ground state was supposed to travel in a circle of radius 5×10^{-11} m, held in orbit by the Coulomb attraction of the proton. According to classical electrodynamics, this electron should radiate, and hence spiral into the nucleus. Show that $v \ll c$ for most of the trip (so you can use the Larmor formula), and calculate the lifespan of Bohr's atom. (Assume each revolution is essentially circular.) [Griffiths's Electrodynamics: Page 487, Question 11.14] (30 Calories)

1.2

Griffiths's Quantum Mechanics: Page 20, Problem 1.9. (30 Calories)

1.3

Griffiths's Quantum Mechanics: Page 22, Problem 1.17. (30 Calories)

1.4

Griffiths's Quantum Mechanics: Page 23, Problem 1.18. (30 Calories)

1.5 *

Griffiths's Quantum Mechanics: Page 22, Problem 1.15. (30 Calories)

1.6 *

The Hamiltonian of a 1-dimensional quantum harmonic oscillator is

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

Its eigen wave functions $\psi_n(x)$ and the corresponding eigen energy E_n are

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{\lambda}{\pi}\right)^{1/4}\exp\left(-\frac{1}{2}\lambda x^2\right)H_n\left(\sqrt{\lambda}x\right)$$

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

where $\lambda = m\omega/\hbar$, $H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$ are the Hermite polynomials.

1. For $n = 0, 1, 2$, find out and sketch the wave functions $\psi_n(x)$ and the square of the wave functions $|\psi_n(x)|^2$ on top of the potential. (10 Calories)

2. For $n = 0, 1, 2$, check the orthogonality of $\psi_n(x)$ by explicit integration. (15 Calories)

3. For $n = 0, 1, 2$, compute $\langle x\rangle$, $\langle x^2\rangle$, σ_x, $\langle p\rangle$, $\langle p^2\rangle$, σ_p. Compute the uncertainty relation quantity $\sigma_x\sigma_p$. Check whether the uncertainty relation is satisfied. Which state is closest to the uncertainty limit? (15 Calories)

4. For $n = 0, 1, 2$, compute the expectation values of the kinetic energy $\langle T\rangle$ and the potential energy $\langle V\rangle$. (15 Calories)

5. If the particle starts out in the initial state

$$\Psi(x, 0) = A\left[3\psi_0(x) + 4\psi_1(x)\right]$$

   what is A? what is $\Psi(x, t)$? If we measure the energy of this particle, what values may we get and with what probabilities? What's the expectation value of $\langle H\rangle$? (15 Calories)

2 Homework 2

Readings:

Chapter 1 and 2, D. J. Griffiths, *Introduction to Quantum Mechanics*, 2nd edition, Pearson Prentice Hall (2004).

2.1

Consider a particle in a one dimensional infinite square well potential:

$$V(x) = \begin{cases} 0, & 0 \le x \le a \\ \infty, & \text{otherwise} \end{cases}$$

1. Compute the eigen energies and the eigen state wave functions. (20 Calories)
2. Compute $\langle x \rangle$, $\langle x^2 \rangle$, σ_x, $\langle p \rangle$, $\langle p^2 \rangle$, σ_p for the n-th eigen state. Compute the uncertainty relation quantity $\sigma_x \sigma_p$. Check whether the uncertainty relation is satisfied. Which state is closest to the uncertainty limit? (20 Calories)
3. If the initial wave function is

$$\Psi(x,0) = \begin{cases} Ax, & 0 \le x \le \frac{a}{2} \\ A(a-x), & \frac{a}{2} \le x \le a \\ 0, & \text{otherwise} \end{cases}$$

   for some constant A.

   (a) Sketch $\Psi(x,0)$. Determine the constant A. (10 Calories)
   (b) Compute $\Psi(x,t)$. (20 Calories)
   (c) If we perform a measurement of the energy, what values may we get and with what probabilities? What's the expectation value? (10 Calories)

2.2

Griffiths's Quantum Mechanics: Page 38, Problem 2.5. (30 Calories)

2.3

Griffiths's Quantum Mechanics: Page 84, Problem 2.35. (30 Calories)

2.4 *

In atomic and nuclear physics, many systems can be described by pseudo-potentials, such as the Fermi pseudo-potential $V(r) = \frac{2\pi\hbar^2}{m} b\delta(r)$ used in describing neutron scattering processes, where b is the bound scattering length. Let's consider a particle with mass m in the 1-dimensional δ-potential

$$V(x) = -V_0 \delta(x)$$

where $V_0 > 0$ is the strength of the potential with the dimension of [EL].

1. Compute the bound state ($E < 0$) wave function and the corresponding eigen energy level(s). (30 Calories)

2. Compute the probability current of the wave function in Part 1. Explain the physical meaning of the result. (5 Calories)

3. Compute the transmitted and reflected wave functions of an incoming plane wave $\psi_i(x) = Ae^{ikx}$ ($E > 0$, scattering state). (30 Calories)

4. Compute the probability current on both sides of the potential in Part 3. Explain the physical meaning of the result. (10 Calories)

5. Compute the transmission coefficient T and reflection coefficient R in Part 3. (10 Calories)

6. Perform dimensional analysis on the eigen energy computed in Part 1, and the transmission and reflection coefficients computed in Part 5. (10 Calories)

7. What's the asymptotic behavior of T and R if the potential is very deep, i.e. $V_0 \to \infty$? Explain the physical meaning. (5 Calories)

3 Homework 3

Readings:

Chapter 1 and 2, D. J. Griffiths, *Introduction to Quantum Mechanics*, 2nd edition, Pearson Prentice Hall (2004).

3.1

1. From the continuity equation
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$
derive the probability current
$$\mathbf{J} = \frac{i\hbar}{2m}(\Psi \nabla \Psi^* - \Psi^* \nabla \Psi) = \frac{\hbar}{m}\text{Im}\{\Psi^* \nabla \Psi\}$$
(10 Calories)

2. Compute the probability current for the plain wave $\Psi(x,t) = e^{i(kx-\omega t)}$. (10 Calories)

3. Compute the probability current for the spherical wave $\Psi(\mathbf{r},t) = e^{i(\mathbf{kr}-\omega t)}/\mathbf{r}$. (10 Calories)

3.2

Griffiths's Quantum Mechanics: Page 85, Problem 2.36. (30 Calories)

3.3

Griffiths's Quantum Mechanics: Page 83, Problem 2.33. (30 Calories)

3.4

Macroscopic Quantum World The Planck constant (reduced) $\hbar \approx 1 \times 10^{-34} J \cdot s$ plays a fundamental role in quantum mechanics. Imagine that one day you are teletransported to another universe, where the reduced Planck constant is 10^{34} times larger, i.e. $\hbar \approx 1 J \cdot s$. Let's picture what strange phenomena you would expect.

1. (*) First, derive the uncertainty principle
$$\sigma_A \sigma_B \geq \frac{1}{2} |\langle [A, B] \rangle|$$
and state its the significance. (10 Calories)

2. Suppose you have a jar of candies in this quantum world. When you open the jar, estimate the escape velocities of the candies. Do you need to be careful when opening the jar? (A typical weight of a candy is 1 gram. A typical size of a jar is 10 cm.) (10 Calories)

3. Use your imagination, describe another two strange phenomena you would expect in this quantum world. Be as quantitative as possible. (20 Calories)

3.5 *

Let's reconsider the 1-dimensional quantum harmonic oscillator

$$H = T + V = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

using the ladder operators

$$a = \sqrt{\frac{m\omega}{2\hbar}}(x + \frac{ip}{m\omega})$$
$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(x - \frac{ip}{m\omega})$$

1. Solve x and p in terms of a and $a\dagger$. Show that the Hamiltonian operator can be written in terms of the ladder operators $H = (N + \frac{1}{2})\hbar\omega$, where $N = a^\dagger a$. (10 Calories)
2. Compute $[a, a^\dagger]$, $[N, a]$, and $[N, a^\dagger]$, where $N = a^\dagger a$. (15 Calories)
3. Show that the energy eigenstates are $|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$ and the energy eigenvalues are $E_n = (n + \frac{1}{2})\hbar\omega$. (20 Calories)
4. Sketch the wave functions of the first three eigen states (n = 0, 1, and 2) in a harmonic potential. (15 Calories)
5. For the n^{th} energy eigenstate $|n\rangle$, compute the expectation values of $\langle n|x|n\rangle$, $\langle n|x^2|n\rangle$, $\langle n|p|n\rangle$, $\langle n|p^2|n\rangle$, and the uncertainty relation quantity $\sigma_x\sigma_p$. (10 Calories)
6. For the n^{th} energy eigenstate $|n\rangle$, compute the expectation values of the kinetic energy $\langle n|T|n\rangle$ and the potential energy $\langle n|V|n\rangle$. (10 Calories)
7. If the particle starts out in the initial state

$$|\Psi(t = 0)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

   what is its state at time t? If we perform measurements of the kinetic energy and potential energy of the system at time t, what do we get? Do they depend on time? (10 Calories)
8. Give one physical example of such quantum harmonic oscillators. Hint: one of such examples is a nuclear material. (10 Calories)